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Fractional quantum hall effect on the 2-sphere: a state-counting analysis

Kåre Olaussen and Roger Sollie†

Gruppe for teoretisk fysikk, Universitetet i Trondheim, Norges Tekniske Høgskole, N-7034 Trondheim-NTH, Norway

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Abstract. We consider a system of electrons confined to the surface of a (large) 2-sphere around a (strong) magnetic monopole, and interacting through a general rotational invariant potential. The Hilbert space for this system can be decomposed into subspaces with definite quantum numbers J_z, $J(J+1)$, and N for respectively the rotation ($\boldsymbol{J}_z$, $\boldsymbol{J}^2$) and particle number $\boldsymbol{N}$ operators. We study the systematics of the dimensions of these subspaces, which reveals many interesting patterns at low N. However, as N increases an overall smooth behaviour is approached, and we calculate this asymptotic behaviour as $N \to \infty$. As a by-product of our analysis we find the exact density of states for the case when the electrons interact via a pair interaction $V(\hat{r}_i, \hat{r}_j) = -v[1 - \cos \vartheta_{ij}]$, where ϑ_{ij} is the spherical angle between the pair.

1. Introduction

A charged particle in a homogeneous magnetic field is probably one of the simplest systems exhibiting an 'anomalous' realisation of a symmetry. This system is invariant under the group E(2) of Euclidean motions in the plane orthogonal to the magnetic field, and we may use the Noether procedure to construct the associated conserved quantities. With the Lagrangian $L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 - e\dot{\boldsymbol{x}}\boldsymbol{A}$, where $A_x = -\frac{1}{2}yB + \partial_x\Lambda$, $A_y = \frac{1}{2}xB + \partial_y\Lambda$ (i.e. with the magnetic field pointing in the z direction, $\boldsymbol{B} = B\hat{\boldsymbol{e}}_z$) these are

$$\begin{aligned} J_z &= xp_y - yp_x + ex\partial_y\Lambda - ey\partial_x\Lambda \\ P_x &= p_x + \tfrac{1}{2}eyB + e\partial_x\Lambda \\ P_y &= p_y - \tfrac{1}{2}exB + e\partial_y\Lambda \end{aligned} \tag{1}$$

coming respectively from invariance under rotations around the z axis and translations in the x and y directions. In all these expressions $\Lambda = \Lambda(x, y)$ is an arbitrary gauge function. In the canonical formalism the conserved quantities above will act as generators of the symmetry transformations from which they were derived. However, in this case, their Poisson brackets do not quite fulfil the Lie algebra of the two-dimensional Euclidean group. Instead

$$\{P_x, P_y\} = eB \qquad \{J_z, P_x\} = P_y \qquad \{J_z, P_y\} = -P_x. \tag{2}$$

For a system of N particles of equal charge e the first 'anomalous' bracket is replaced by eBN. (The reason for the discrepancy is that the generators act on the whole 4D phase space, not only on the 2D Euclidean plane.)

† Present address: Department of Physics, Temple University, Philadelphia, PA 19122, USA.

Thus, after quantisation, it is not possible to choose simultaneous eigenstates for P_x and P_y, since these operators do not commute with each other. Therefore, when analysing a finite-size version of the model, e.g. for the quantum Hall effect (QHE), it is of no particular advantage to maximise the translational symmetries by imposing periodic boundary conditions as usual. Instead, it seems computationally simpler to map the xy plane onto the surface of a 2-sphere [1], thus modelling the system by a huge magnetic monopole [2]. In this process the Euclidean symmetry of the xy plane is replaced by the rotational symmetry of the sphere, with the latter being realised in the standard way. In the following we shall not consider motion in the z direction, assuming it to be frozen, as is usual in discussions of the QHE.

Taking the magnetic monopole to carry q units of magnetic charge, where q is integer or half integer (to avoid unnecessary cluttering of our formulae we assume q to be positive), the radius R of the 2-sphere must be chosen to satisfy

$$4\pi R^2 B = 4\pi q\hbar/e \equiv 2q\Phi_0 \tag{3}$$

with Φ_0 the elementary flux quantum. That is, $R = q^{1/2} l_B$, where $l_B \equiv (\hbar/eB)^{1/2}$ is the magnetic length.

Quantisation of the kinetic energy leads to the energy levels [3]

$$E_L = \hbar\omega_c[L + \tfrac{1}{2} + L(L+1)/2q] \qquad L = 0, 1, \ldots, \tag{4}$$

where $\omega_c = eB/m$ is the cyclotron frequency. The states in the Lth (Landau) level carry angular momentum $J = q + L$, and thus have a degeneracy $D_L = 1 + 2q + 2L$.

The one-particle eigenfunctions can be chosen as the monopole harmonics [4]. Some of their useful properties are given by Wu and Yang [5]†. In the lowest Landau level these monopole harmonics are related to the singular gauge eigenfunctions, $h_m(\theta, \varphi) = \cos^{q+m}(\frac{1}{2}\theta)\sin^{q-m}(\frac{1}{2}\theta)e^{im\varphi}$, used by Haldane and Rezayi [6], by

$$Y_{qm}^{(-q)}(\theta, \varphi) = \left(\frac{2q+1}{4\pi}\right)^{1/2} \binom{2q}{q+m}^{1/2} h_m(\theta, \varphi) \times \begin{cases} e^{-iq\varphi} & \text{on Northern hemisphere} \\ e^{+iq\varphi} & \text{on Southern hemisphere.} \end{cases} \tag{5}$$

These expressions show that, when $q \to \infty$, a (negative charge) particle of angular momentum $J_z = \hbar m$ will be localised to the region where $\cos\theta \simeq m/q$.

Let us further verify that the commutation relations for the rotation group will reduce to those found for (the 'anomalously' realised) E(2) when $q \to \infty$. Consider a particle in the vicinity of the North Pole. By a simple geometric consideration we see that $P_x = R^{-1}J_y = q^{-1/2} l_B^{-1} J_y$ and $P_y = -R^{-1}J_x = -q^{-1/2} l_B^{-1} J_x$ are the appropriate generators of translations in this region. Their Poisson bracket thus becomes

$$\{P_x, P_y\} = -\frac{i}{\hbar}[P_x, P_y] = \frac{i}{\hbar q l_B^2}[J_y, J_x] = \frac{1}{q l_B^2} J_z = eB \tag{6}$$

in agreement with the direct computation (2). In the last identity above we have used that a particle close to the North Pole has $J_z \simeq \hbar q$.

The system we are considering may be regarded as a model for electrons confined to an essentially two-dimensional GaAs-Ga_xAl_{1-x} As heterostructure (or Si-MOSFET inversion layer) in a strong magnetic field. Low-temperature experiments have revealed

† Since we have assumed the magnetic field to be positive, while the electron has a negative charge $-e$, our one-particle eigenfunctions are $Y_{Jm}^{(-q)}$, $J = q, q+1, \ldots$.

the spectacular phenomenon of the fractional quantum Hall effect (FQHE) in clean samples of such systems [7]. These experiments have been interpreted as a distinct affinity for the systems to stay at certain rational filling factors. Using arguments of symmetry and particle statistics alone, Laughlin [8] found Jastrow-type wavefunctions which appear to be very good approximations to the ground state at filling factors $\nu = 1/m$, when $m = 2k+1$ is a small odd integer. Precisely such factors are among the experimentally most distinguished ones. And, interestingly enough, it is not possible to make a simple generalisation of the Laughlin states to arbitrary filling factors.

Although there is little doubt that the Laughlin–Jastrow wavefunctions represent the FQHE ground states very well in the appropriate cases, the fact that one cannot find simple representations of the ground states except at particular filling factors is somewhat unsatisfactory as an explanation for the FQHE. Why should Nature care about whether it can be described in terms of a simple wavefunction or not?

However, since symmetry and particle statistics were the essential ingredients in the derivation of the Laughlin states (apart from the Jastrow ansatz), it is interesting to investigate the consequences of these restrictions alone. This is the purpose of the present paper, where we analyse how the Hilbert space for N identical fermions, all being in the lowest Landau level of the field of a magnetic monopole with q units of magnetic charge, split into irreducible representations (i.e. subspaces of definite angular momentum $J^2 = J(J+1)$) of the rotation group. The rotation symmetry appears to be the only one which may be present irrespective of the type of interaction. Our hope was that this analysis would reveal some interesting patterns in the number of times each J occur, as functions of the filling factor ν. Our results are that, after a rather interesting initial structure at low particle numbers (which correlates well with the stability pattern of the ground state energies), an essentially smooth behaviour is obtained when $q \to \infty$ with J and ν fixed. With the *a priori* knowledge that the asymptotic behaviour is smooth, this behaviour is straightforward to compute analytically (together with the approach to asymptotica). We do that, and obtain perfect agreement with the results of the exact numerical calculation.

The rest of this paper is organised as follows. In section 2 we use the character formula to derive a recursion relation for the required dimensions $D(N, J; q)$, and perform an exact numerical computation of the Ds up to fairly large values of N and q. In section 3 we derive analytic expressions for the asymptotic behaviour of the Ds as $q \to \infty$ with $\nu \equiv N/(2q+1)$ and $\sigma \equiv 2J/q(q+1)$ fixed, and compare with the exact results. We close our discussion in section 4 with a few comments.

2. Exact state counting from the character formula

Since the Hamiltonian of the system commutes with the particle number $\boldsymbol{N}$ and rotation $\boldsymbol{J}$ operators, it can be made block diagonal, with the blocks being labelled by the eigenvalues N, J_z, $J(J+1)$ of the operators $\boldsymbol{N}$, $\boldsymbol{J}_z$, $\boldsymbol{J}^2$. The linear dimension of each block, $D(N, J; q)$, is equal to the number of times the spin-J representation of the rotation group (actually SU(2), if q is half integer) occurs in the subspace of fixed N. Finding $D(N, J; q)$ is thus a problem of coupling spin-q altogether N times, and counting how many times the result decomposes into each irreducible representation. However, since we must take into account the Pauli principle, only representations which are totally antisymmetric in the N factors are allowed. This is automatically taken care of if we employ second quantisation.

In second quantised language, the dimension $D(N, J; q)$ can be found from the character formula

$$\chi(\theta, \phi; q) = \mathrm{Tr}\{e^{i\theta N} e^{i\phi J_z}\} = \sum_{N,J} D(N, J; q)\, e^{iN\theta} \chi^{(J)}(\phi) \tag{7}$$

where $\chi^{(J)}$ is the character for the spin-J representation of SU(2),

$$\chi^{(J)}(\phi) = \sum_{m=-J}^{J} e^{im\phi} = \frac{\sin[(J+\frac{1}{2})\phi]}{\sin(\frac{1}{2}\phi)} \tag{8}$$

and the operators N and J_z are given by $N = \Sigma_m a_m^\dagger a_m$, $J_z = \Sigma_m m a_m^\dagger a_m$, when an appropriate representation for the fermion operators is chosen.

In this representation the character $\chi(\theta, \phi; q)$ is easy to find:

$$\chi(\theta, \phi; q) = \prod_{m=-q}^{q} [1 + e^{i(\theta + m\phi)}]. \tag{9}$$

Using the orthogonality properties of the characters $\chi^{(J)}$, an explicit expression for the $D(N, J; q)$ is then given by the integral formula

$$D(N, J; q) = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} d\theta\, e^{-iN\theta} \int_{-\pi}^{\pi} d\phi\, \sin^2(\tfrac{1}{2}\phi) \chi^{(J)}(\phi) \chi(\theta, \phi; q)$$

but this is not very practical for obtaining explicit results. It is better to utilise the recursion relation

$$\chi(\theta, \phi; q) = [1 + 2\, e^{i\theta} \cos(q\phi) + e^{2i\theta}] \chi(\theta, \phi; q-1),$$

and employ the formula

$$2\cos(q\phi)\chi^{(J)}(\phi) = \chi^{(J+q)}(\phi) + \mathrm{sgn}(J-q)\chi^{(|J-q|)}(\phi)$$

to find a recursion relation in q for $D(N, J; q)$:

$$\begin{aligned} D(N, J; q) = {} & D(N, J; q-1) + D(N-1, J+q; q-1) + D(N-1, J-q; q-1) \\ & - D(N-1, q-J; q-1) + D(N-2, J; q-1). \end{aligned} \tag{10}$$

Here $D(N, J; q-1)$ is defined to be zero if $N < 0$ or $J < 0$. The $D(N, J; q)$ are also zero when $J > J_{\max} = \frac{1}{2}N(2q+1-N)$. The recursion relation of (10) thus involves only a finite number of non-zero quantities for each q. The starting values may be read out of the character expression for $q = 0$ (or $q = \frac{1}{2}$):

$$\begin{aligned} & D(0,0;0) = D(1,0;0) = 1 && \text{if } q \text{ is integer} \\ & D(0,0;\tfrac{1}{2}) = D(1,\tfrac{1}{2};\tfrac{1}{2}) = D(2,0;\tfrac{1}{2}) = 1 && \text{if } q \text{ is half integer.} \end{aligned} \tag{11}$$

It is now straightforward to make an exact numerical computation of all the required dimensions up to fairly high values of q. As a consistency check on the computation, the relation

$$\sum_J (2J+1) D(N, J; q) = \binom{2q+1}{N} \tag{12}$$

may be tested. From the point of view of numerical diagonalisation of the Hamiltonian it is also of interest to compare the dimensions $D(N, J; q)$ with the dimensions $C(N, J; q)$ obtained if only the quantum numbers N and J_z are fixed. The latter are the dimensions of the so-called configuration interaction matrices (which are the

Hamiltonian matrices that can be constructed in a fairly straightforward manner in these systems). We have the relation

$$C(N, M; q) = \sum_{J \geq M} D(N, J; q). \tag{13}$$

At low particle numbers $D(N, J; q)$ reveals some rather interesting behaviour as a function of q.

(i) For $N=2$ we find $D=1$ when $2q+J$ is an even integer, and $D=0$ otherwise. This is a reflection of the well known fact that two equal spins q can couple antisymmetrically only to an odd angular momentum when q is integer, and only to an even angular momentum when q is half integer. Still, it is worth noticing that this is in perfect agreement with the condition for having a Laughlin wavefunction on the sphere [1]

$$q = \tfrac{1}{2}m(N-1) \tag{14}$$

where $m = 2k+1$ is an odd integer. For $N=2$ this predicts a Laughlin state for every half integer, i.e. all the $J=0$ states possible.

(ii) For $N=3$ this pattern is maintained. A (single) $J=0$ state exists if and only if q is an odd integer, $q=2k+1$, in agreement with (14). That is, all the possible $N=3$, $J=0$ states are Laughlin states. For other values of J the pattern is more complicated, revealing a period-6 systematic variation, and being dependent on whether $J \leq q$ or not. We summarise the complete pattern in table 1 for integer q.

Table 1. The dimensions $D(N=3, J; q)$ of the subspaces for all integer values of J and q. Here $k=0, 1, \ldots$.

	$D(3, J; q)$			
	q odd		q even	
J	$0 \leq J \leq q$	$q+1 \leq J \leq 3q-3$	$0 \leq J \leq q$	$q+1 \leq J \leq 3q-3$
$6k$	$2k+1$	$\frac{1}{2}q+\frac{1}{2}-k$	$2k$	$\frac{1}{2}q-k$
$6k+1$	$2k$	$\frac{1}{2}q-\frac{1}{2}-k$	$2k+1$	$\frac{1}{2}q-k$
$6k+2$	$2k+1$	$\frac{1}{2}q-\frac{1}{2}-k$	$2k$	$\frac{1}{2}q-k-1$
$6k+3$	$2k+1$	$\frac{1}{2}q-\frac{1}{2}-k$	$2k+2$	$\frac{1}{2}q-k$
$6k+4$	$2k+2$	$\frac{1}{2}q-\frac{1}{2}-k$	$2k+1$	$\frac{1}{2}q-k-1$
$6k+5$	$2k+1$	$\frac{1}{2}q-\frac{3}{2}-k$	$2k+2$	$\frac{1}{2}q-k-1$

(iii) The number of $N=4$, $J=0$ states must be equal to $D(N=3, J=q; q)$, and can thus be read out of table 1 for integer q. The complete pattern is that

$$D(N=4, J=0; q) = k, k-1, k, k, k, k, \ldots \tag{15}$$

when

$$2q = 6k-3, 6k-2, 6k-1, 6k, 6k+1, 6k+2, \ldots.$$

Here $k = 1, 2, \ldots$. We note that the number of states is increased by one at $q=3/2$, $9/2, \ldots, (6k+3)/2, \ldots$. This again agrees with the sequence of (14), and it is natural to identify the new state which appears with a Laughlin state. Since this state temporarily disappears when q is further increased by $\frac{1}{2}$, one might expect these states to have an additional stability with respect to changes in the filling factor.

(iv) Up to $N=4$ we have found a one-to-one correspondence between the sequence of Laughlin states and the number of $J=0$ states. Unfortunately this correspondence cannot easily be seen at $N=5$, although there still is some regular periodic structure in $D(N=5, J=0; q)$. This is simplest to describe by defining $\Delta_D(q) = D(5,0;q) - D(5,0;q-2)$, which varies as

$$\Delta_D(q) = k, k-1, k, k, k, k, \ldots \tag{16}$$

when

$$q = 12k-10, 12k-8, 12k-6, 12k-4, 12k-2, 12k, \ldots.$$

Here $k=1,2,\ldots$. This structure only involves the qs at even integers. In addition, there is the relation $D(5,0;q+9) = D(5,0,q)$. One may ponder if this originates in some kind of symmetry, operating between different q-sectors.

(v) Apart from a systematic difference between integer and half integer q, at $N=6$ it seems that $D(N,0;q)$ begin to approach a rather smooth behaviour. The second-order difference, $D(6,0;q+1) - 2D(6,0;q) + D(6,0;q-1)$, turns out to be rather small, but it does not reveal any simple pattern. However, analogous to the $N=5$ case there is a relation between the dimensions at integer and half integer q: $D(6,0,q+15/2) = D(6,0,q)$.

As N increases it seems that $D(N,0;q)$ approaches a completely smooth behaviour. The systematic difference between odd and even q (when N is odd), and between integer and half integer q (when N is even) also disappears as N increases. This is demonstrated in figure 1, where we show how $\ln D(N,J;q)/q$ becomes a smooth function of the scaled variables $\nu \equiv N/(2q+1)$ and $\sigma = 2J/q(q+1)$ as we approach

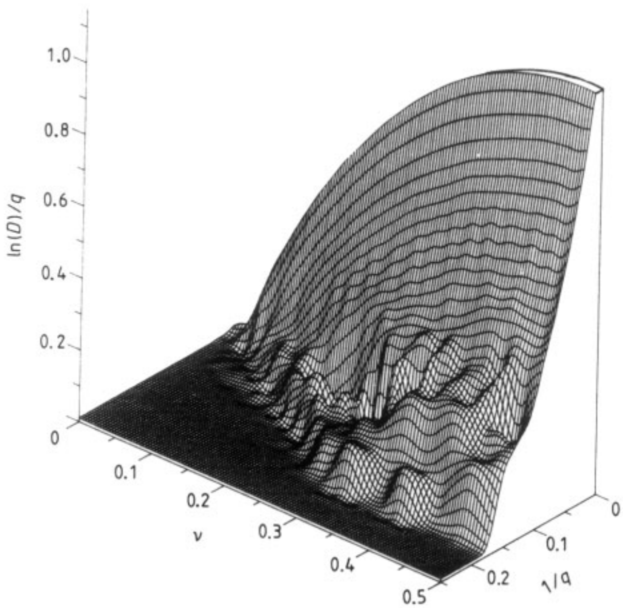

Figure 1. The dimension of the subspaces as a function of $1/q$ and ν. Due to scaling we actually plot the logarithm of D divided by q. The small oscillations for intermediate values of q is an artifact caused by the graphics routine (cf the smooth behaviour in figures 2 and 3).

the $q=\infty$ limit. (Only the case of $\sigma=0$ is shown.) In tables 2 and 3 we show results for $C(N,0;q)$ and $D(N,0;q)$ for some low values of N, q. The sequences chosen in table 2 correspond to the spherical analogues of the Laughlin $\frac{1}{3}$ and $\frac{1}{5}$ states, i.e. $q=\frac{1}{2}m(N-1)$, with $m=3$ and 5. The sequences chosen in table 3 correspond to the first two daughters of the $\frac{1}{3}$ states, i.e. $q=\frac{1}{2}m(N-1)\pm\frac{1}{2}(1+N/p)$, with $m=3$ and $p=2$. These two sequences approach filling factors $\nu=\frac{2}{7}$ and $\frac{2}{5}$ as $N\to\infty$. It is worth noticing the large difference in magnitude of the coefficients C and D. This tells us that the matrices one has to diagonalise are much smaller if one is able to exploit the full rotational symmetry. As $q\to\infty$ the ratio C/D behaves like $4q^3\nu(1-\nu)/3$ when $J=0$, and like $q/\lambda(\nu,\sigma)$ when $\sigma\sim J/q^2\neq 0$ (cf the discussion in section 3). Thus, the savings are greatest in the most interesting case of rotational invariant states. It is, however, non-trivial to explicitly construct the Hamiltonian matrices in the subspaces where $\boldsymbol{J}^2$ is diagonal [9].

Table 2. Dimensions $D(N,J=0;q)$ $(C(N,M=0;q))$ of the subspaces diagonalising N, $\boldsymbol{J}^2$ and J_z (N and J_z) for some small values of q and N corresponding to the $m=3$ and 5 Laughlin states. These exist when $q=\frac{1}{2}m(N-1)$ (m odd), and approach a filling factor $\nu=m^{-1}$ as $N\to\infty$.

	$m=3$			$m=5$		
N	q	$D(q)$	$C(q)$	q	$D(q)$	$C(q)$
4	4.5	2	18	7.5	3	86
5	6.0	2	73	10.0	4	649
6	7.5	6	338	12.5	20	5 444
7	9.0	10	1 656	15.0	62	48 417
8	10.5	31	8 512	17.5	365	450 096
9	12.0	84	45 207	20.0	2 082	4 323 349
10	13.5	319	246 448	22.5	14 664	42 611 589
11	15.0	1 160	1 371 535	25.0	106 678	428 774 562
12	16.5	4 498	7 764 392	27.5	833 361	
13	18.0	21 660	44 585 180	30.0	6731 131	

Table 3. Dimensions $D(N,J=0;q)$ $(C(N,M=0;q))$ of the subspaces diagonalising N, $\boldsymbol{J}^2$ and J_z (N and J_z) for some small values of q and N corresponding to the $m=3$, $p=\pm2$ hierarchy states. These exist when $q=\frac{1}{2}m(N-1)\pm\frac{1}{2}(1+N/p)$ (m odd, p even and dividing N), and approach the filling factors $\nu=p/(mp\pm1)$ as $N\to\infty$.

	$m=3, p=-2$			$m=3, p=2$		
3	q	$D(q)$	$C(q)$	q	$D(q)$	$C(q)$
4	3.0	1	5	6.0	2	43
6	5.5	3	58	9.5	10	1 242
8	8.0	8	910	13.0	80	46 029
10	10.5	52	16 660	16.5	1 429	1 943 488
12	13.0	320	332 578	20.0	35 123	89 008 530
14	15.5	5 034	7 040 196	23.5	1060 615	
16	18.0	70 180	15 548 450	27.0		
18	20.5	1099 354		30.5		

3. Asymptotic behaviour of the number of states

As demonstrated by figure 1, exact calculation of the $D(N, J; q)$ from the recursion relation (10) reveals a smooth behaviour when $q \to \infty$. We may thus attempt to calculate this asymptotic behaviour directly. To this end we consider the quantity

$$Z(\mu, \lambda) = \mathrm{Tr}\{e^{-(\mu N + \lambda J_z/q)}\} \tag{17}$$

which on the one hand can be expressed in terms of the quantities $C(N, M; q)$, and on the other hand can be evaluated directly as in (9):

$$\begin{aligned} Z(\mu, \lambda) &= \sum_{N=0}^{2q+1} \sum_{M=-q(q+1)/2}^{q(q+1)/2} C(N; M; q)\, e^{-(\mu N + \lambda M/q)} \\ &= \exp\left\{ \sum_{m=-q}^{q} \ln[1 + e^{-(\mu + m\lambda/q)}] \right\}. \end{aligned} \tag{18}$$

Evaluating the last sum with the help of the Euler–Maclaurin summation formula we find as $q \to \infty$:

$$Z(\mu, \lambda) = \exp\{qA(\mu, \lambda) + B(\mu, \lambda) + O(q^{-1})\} \tag{19}$$

where

$$\begin{aligned} A(\mu, \lambda) &= \int_{-1}^{1} dx \ln[1 + e^{-(\mu + \lambda x)}] \\ B(\mu, \lambda) &= \tfrac{1}{2}\ln[(1 + e^{-(\mu - \lambda)})(1 + e^{-(\mu + \lambda)})]. \end{aligned} \tag{20}$$

We now make the ansatz

$$C(N, M; q) = \exp\{qa(\nu, \sigma) + b(\nu, \sigma) \ln q + c(\nu, \sigma) + O(q^{-1})\} \tag{21}$$

when $q \to \infty$. Here we have introduced scaled variables

$$\nu = \frac{N}{2q+1} \text{ (filling factor)} \qquad \sigma = \frac{2M}{q(q+1)} \tag{22}$$

and $a(\nu, \sigma)$, $b(\nu, \sigma)$, and $c(\nu, \sigma)$ are functions to be determined. We insert this ansatz into (18), and approximate the sum over N, M by an integral. Evaluating this integral as $q \to \infty$ gives

$$\begin{aligned} Z(\mu, \lambda) = \pi(2q+1)(q+1) \det{}^{-1/2}(-\partial^2 a) \exp\{q[a(\nu, \sigma) - 2\mu\nu - \tfrac{1}{2}\lambda\sigma] \\ + b(\nu, \sigma) \ln q + [c(\nu, \sigma) - \mu\nu - \tfrac{1}{2}\lambda\sigma]\} \end{aligned} \tag{23}$$

where $(\partial^2 a)$ is the 2×2 matrix of second-order partial derivatives (cf (28)). Here ν, σ are to be viewed as functions of μ, λ, determined from the equations

$$\frac{\partial a}{\partial \nu} = 2\mu \qquad \frac{\partial a}{\partial \sigma} = \tfrac{1}{2}\lambda. \tag{24}$$

By comparing (19) and (23) we realise that $A(\mu, \lambda)$ and $a(\nu, \sigma)$ are related by Legendre transforms:

$$a(\nu, \sigma) = A(\mu, \lambda) + 2\mu\nu + \tfrac{1}{2}\lambda\sigma \tag{25}$$

with

$$\nu = -\frac{1}{2}\frac{\partial A}{\partial \mu} \qquad \sigma = -2\frac{\partial A}{\partial \lambda}. \tag{26}$$

Further, with the above relations between (ν, σ) and (μ, λ),

$$b(\nu, \lambda) = -2$$
$$c(\nu, \lambda) = B(\mu, \lambda) + \mu\nu + \tfrac{1}{2}\lambda\sigma - \tfrac{1}{2}\ln[\det(\partial^2 A)] - \ln 2\pi. \tag{27}$$

Here we have made use of the fact that the matrices $(\partial^2 a)$ and $(\partial^2 A)$ are essentially inverse of each other when a and A are related by Legendre transforms:

$$(\partial^2 a) \equiv \begin{pmatrix} \frac{1}{2}\frac{\partial^2 a}{\partial \nu^2} & \frac{\partial^2 a}{\partial \nu\, \partial \sigma} \\ \frac{\partial^2 a}{\partial \sigma\, \partial \nu} & 2\frac{\partial^2 a}{\partial \sigma^2} \end{pmatrix} = -\begin{pmatrix} \frac{1}{2}\frac{\partial^2 A}{\partial \mu^2} & \frac{\partial^2 A}{\partial \mu\, \partial \lambda} \\ \frac{\partial^2 A}{\partial \lambda\, \partial \mu} & 2\frac{\partial^2 A}{\partial \lambda^2} \end{pmatrix}^{-1} = -(\partial^2 A)^{-1}. \tag{28}$$

We have now collected all the formulae necessary to compute the asymptotic behaviour of $C(N, M; q)$ as $q \to \infty$. This is in turn related to the asymptotic behaviour of $D(N, J; q)$ by

$$D(N, J; q) = C(N, J; q) - C(N, J+1; q) = -\left\{\frac{\partial}{\partial J} + \frac{1}{2}\frac{\partial^2}{\partial J^2} + \ldots\right\} C(N, J; q)$$
$$= -\left\{\frac{2}{q(q+1)}\frac{\partial}{\partial \sigma} + \frac{2}{q^2(q+1)^2}\frac{\partial^2}{\partial \sigma^2} + \ldots\right\}$$
$$\times \exp\{qa(\nu, \sigma) - 2\ln q + c(\nu, \sigma) + \ldots\}. \tag{29}$$

We have to expand to second order in the derivative here because the first derivative vanishes when $J = 0$, since $C(N, J; q)$ is an even function of J. For this reason the ratio $D(N, J; q)/C(N, J; q)$ scales like q^{-3} when $J \simeq 0$, and like q^{-1} otherwise (cf the discussion below (13)).

Since the integral in (20) defining $A(\mu, \lambda)$ cannot be calculated in closed form, it is not possible in general to give completely explicit expressions for the asymptotic behaviour of D. However, some analytic simplification is possible because we can evaluate $\partial A/\partial \mu$, and eliminate μ in favour of ν from (26): $e^{\mu} = \sinh(1-\nu)/\sinh(\nu)$. The remaining analysis must then be performed numerically. This we have done in various cases, and verified that the expressions found reproduce the large-q behaviour calculated directly from the recursion relation (10). Further, in the most interesting case of rotational symmetry, $J = \sigma = 0$, explicit expressions are available, since in this case also $\lambda = 0$. We find $A(\mu, 0) = 2\ln(1 + e^{-\mu})$, $\nu = (1 + e^{\mu})^{-1}$, i.e. $e^{\mu} = (1-\nu)/\nu$, and thus

$$a(\nu, 0) = -2[\nu \ln \nu + (1-\nu)\ln(1-\nu)]. \tag{30}$$

To evaluate $c(\nu, 0)$ we first calculate $A_{\mu\mu}(\mu, 0) = 2e^{\mu}/(e^{\mu}+1)^2 = 2\nu(1-\nu)$, $A_{\mu\lambda}(\mu, 0) = 0$, and $A_{\lambda\lambda}(\mu, 0) = 2\nu(1-\nu)/3$, and thus find

$$c(\nu, 0) = -[(1+\nu)\ln \nu + (2-\nu)\ln(1-\nu) + \ln(4\pi/\sqrt{3})]. \tag{31}$$

Collecting terms, the leading-order behaviour for $C(N, 0; q)$ is found to be

$$C(N, 0; q) \simeq (\sqrt{3}/4\pi q^2) \times \exp\{-2q[\nu \ln \nu + (1-\nu)\ln(1-\nu)]$$
$$-[(1+\nu)\ln \nu + (2-\nu)\ln(1-\nu)]\}. \tag{32}$$

This formula is not quite sufficient to calculate the leading-order behaviour for $D(N, 0; q)$. However, from the general expression (29) the connection is found to be

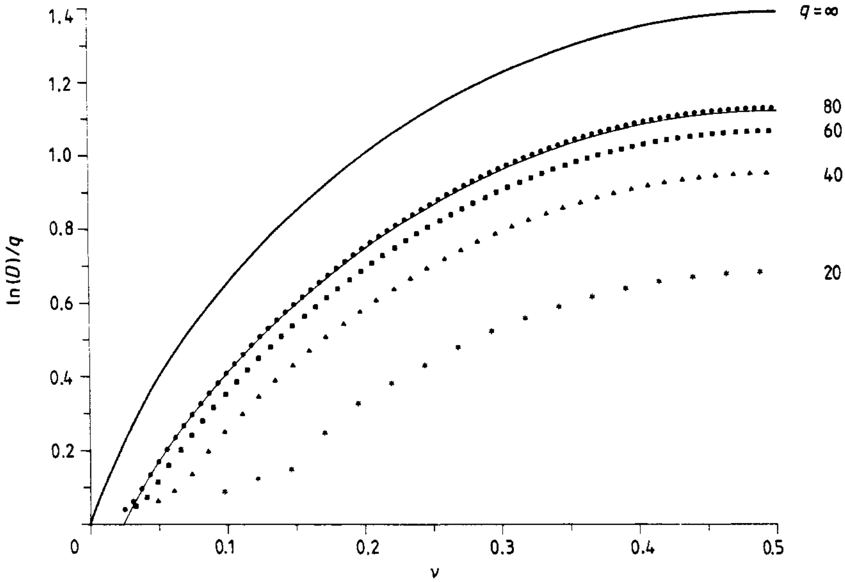

Figure 2. The approach to asymptotia as q increases. The exact values are plotted for $q = 20$, 40, 60 and 80. In addition we show the asymptotic behaviour for $q = 80$ as the full curve close the numerical values, calculated from (33), and also the limit $q = \infty$.

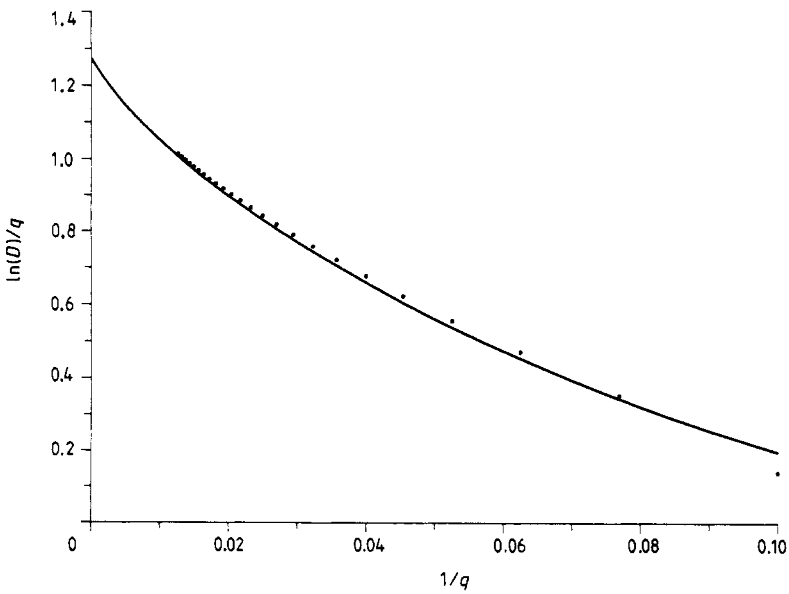

Figure 3. The approach to asymptotia as q increases, for $J = 0$ and $\nu = \frac{1}{3}$. The exact values are compared with the asymptotic behaviour (33).

$D(N, 0; q) = -2q^{-3}(\partial^2 a/\partial\sigma^2)(\nu, 0)C(N, 0; q)$. From (28) we find that $(\partial^2 a/\partial\sigma^2)(\nu, 0) = -3/8\nu(1-\nu)$. Finally

$$D(N, 0; q) \simeq [3\sqrt{3}/16\pi\nu^2(1-\nu)^2 q^5] \exp\{-(2q+1)[\nu \ln \nu + (1-\nu)\ln(1-\nu)]\}. \tag{33}$$

In figures 2 and 3 we show how this analytic formula compares with the numerical results. As can be seen, it is very good except at low values of q. It is also clear that the corrections to the $q = \infty$ result are very important even for rather large values of q. This is somewhat surprising in view of the stable results obtained when computing the ground state energies at low particle numbers, and should perhaps be taken as a warning against putting too much trust in results from numerical calculations on small systems. However, it is possible that the corrections are unusually large in this case, due to the $5\ln(q)/q$ term which is present here.

4. Comments and conclusion

We close our discussion with a few remarks.

(A) In this paper we have used the (in our view) natural definition of filling factor, $\nu = N/(2q+1)$ in the lowest Landau level, $L = 0$. This is slightly different from the one which has been used by Haldane and Rezayi [6] and others [10]. Denoting the latter as ν_H, the relations are

$$\nu = \frac{\nu_H}{1-(1-\nu_H)/N} \qquad \nu_H = \frac{1-1/N}{1-\nu/N}\nu. \tag{34}$$

We have noticed that using our definition of filling factor reduces the finite size dependence of the energy per particle $E_N(\nu)/N$ [11]. However, the spherical analogues of the Laughlin states exist at fixed values of ν_H when N varies, not at fixed values of ν.

(B) When projected onto the lowest Landau level, the operator $\boldsymbol{J}^2$ is related to the Hamiltonian H for a system of particles interacting via a pair potential

$$V(\hat{r}_i, \hat{r}_j) = -v\left[1 - \frac{4\pi}{3}\sum_{m=-1}^{m} Y^*_{1m}(\hat{r}_i)Y_{1m}(\hat{r}_j)\right] = -v[1-\cos\vartheta_{ij}] \tag{35}$$

where ϑ_{ij} is the spherical angle between the two particles. Adding a compensating interaction with a homogeneous background the relation becomes

$$H = \frac{v}{2(q+1)^2}[\boldsymbol{J}^2 + (q+1)N]. \tag{36}$$

Since the spectra of these operators are known, and the degeneracy of each eigenvalue has been computed, we have as a byproduct of our analysis found the complete density of states against energy for this particular model. This is plotted in figure 4 for a selected set of filling factors ν, in the limiting case of $q = \infty$.

(C) The model above is also a natural spherical generalisation of a model in the Euclidean plane with harmonic interactions between the fermions [12]. Since the interaction between distant particles is likely to be screened by the compensating background, we may approximate

$$-v[1-\cos\vartheta_{ij}] \simeq -\tfrac{1}{2}v\vartheta_{ij}^2 = -\frac{1}{2}\frac{v}{ql_B^2}r_{ij}^2$$

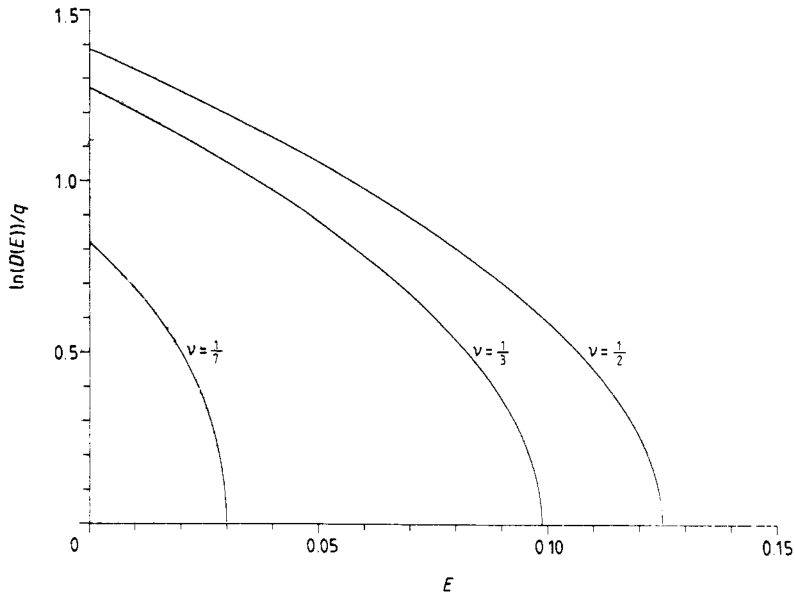

Figure 4. The density of states as a function of energy for the model defined by the pair interaction (35). The model may be viewed as the spherical version of a model with harmonic interactions defined on the Euclidean plane.

as $q \to \infty$. Here r_{ij} is the planar distance between the particles. We should thus make the scaling $v = qE_0$, with E_0 constant as $q \to \infty$. If the ground state is to be rotational invariant we must choose E_0 to be positive. It is clear that this model by itself does not show any sign of the FQHE, but there is left a huge degeneracy (with a finite zero point entropy per particle) of the ground state.

In conclusion, we have analysed the number of states in the 2D many-electron problem in a perpendicular magnetic field, with all electrons restricted to the lowest Landau level. The problem was mapped onto a sphere, and calculations of the number of states were made both exact numerically and asymptotically in the thermodynamic limit. The results show some interesting pattern at low values of N, distinguishing the same filling factors as those selected by the (finite N) Laughlin states on a spherical surface. However, no similar correlation is seen for the hierarchial states, and as N increases a smooth dependency on the filling factors is obtained, with no signs of any particular extraordinary values.

The formalism we use stems from Wu and Yang's solutions of an electron in the field of a magnetic monopole. This leads naturally to another definition of the filling factor of the lowest Landau level, slightly different from the one which appears to have been used by others.

We have also shown that the matrix dimension of the Hamiltonian can be largely reduced if it is possible to exploit the full rotational symmetry of the problem. The savings are particularly large for the states of total angular momentum zero. We have performed a case study of how to perform this symmetry reduction [9].

In what way is this work relevant for understanding the fractional quantum Hall effect? We think the main lesson to be learned is that the results from studies on (small) finite systems must be treated with extreme caution. Structures which can be seen at low particle numbers may be washed away completely in the thermodynamic limit.

Thus, even though the finite size studies by Haldane and Rezayi [6] and Fano *et al* [10] give very strong support for the Laughlin theory of the FQHE, it is our belief that there is still more work to be done and discoveries to be made on this problem.

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